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Some researches on trivariate Lagrange interpolation

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Abstract

In this paper, in order to go a step further research on the problem of trivariate Lagrange interpolation, we pose the concepts of sufficient intersection of algebraic surfaces and Lagrange interpolation along a space algebraic curve, and extend Cayley–Bacharach theorem in algebraic geometry from $\mathbf{R}^2$ to $\mathbf{R}^3$. By using the conclusion of the extended theorem, we deduce a general method of constructing properly posed set of nodes for Lagrange interpolation along a space algebraic curve, and give a series of corollaries for the practical applications. Moreover, we give a new method of constructing properly posed set of nodes for Lagrange interpolation along an algebraic surface, and as a result we make clear the geometrical structure of it.

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1. Introduction

The problem of polynomial interpolation in two variables has been studied by many authors in recent years. In 2002, for example, in [4], Hakopian discussed a bivariate interpolation problem and solved the posedness problem for it. In 2003, Bojanov and Xu discussed polynomial interpolation of two variables based on points that are located on multiple circles in [1], Carnicer and Gasca also studied the problems of bivariate Lagrange interpolation on conics (cubics) and classification of bivariate configurations with

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simple Lagrange interpolation formulae in [2,3], respectively. In 2004, Liang in [6] deduced a general method of constructing properly posed set of nodes for bivariate Lagrange interpolation which generalized some main results in [7,5]. All of the above are very excellent. But in many practical applications (for instance, spherical interpolation, surface reconstruction, scattered data interpolation and fitting in $\mathbf{R}^3$), we often need to deal with the problem of trivariate Lagrange interpolation. So, in this paper, we will lay emphasis on discussing trivariate Lagrange interpolation which is closely related to the interpolation along an algebraic surface and a space algebraic curve.

Let n be an integer, k a nonnegative integer and

$$\binom{n}{k} = \begin{cases} 0 & n<k \\ \dfrac{n!}{(n-k)!k!} & n \geqslant k \end{cases} \quad d_n = \binom{n+3}{3},$$

and $\mathbf{P}_n^{(3)}$ denotes the space of all real trivariate polynomials of total degree $\leqslant n$, i.e.

$$\mathbf{P}_n^{(3)} = \left\{ \sum_{0\leqslant i+j+k\leqslant n} a_{ijk}x^i y^j z^k | a_{ijk} \in \mathbf{R} \right\}.$$

Definition 1. Let $\mathscr{A}=\{Q_i\}_{i=1}^{d_n}$ be a set of d_n distinct points on $\mathbf{R}^3$. Given any set $\{f_i\}_{i=1}^{d_n}$ of real numbers, we seek a polynomial $g(X) \in \mathbf{P}_n^{(3)}$ (where $g(X)=g(x,y,z)$) satisfying

$$g(Q_i) = f_i, \quad i=1,\ldots,d_n. \tag{1}$$

If for any given set $\{f_i\}_{i=1}^{d_n}$ of real numbers there always exists a unique solution for the equation system (1), we call the interpolation problem a properly posed interpolation problem and call the corresponding set $\mathscr{A}=\{Q_i\}_{i=1}^{d_n}$ of nodes a properly posed set of nodes (or PPSN, for short) for $\mathbf{P}_n^{(3)}$.

Theorem A (*Wang and Liang [12]*). *A set $\mathscr{A}=\{Q_i\}_{i=1}^{d_n}$ of nodes is a PPSN for $\mathbf{P}_n^{(3)}$ if and only if $\mathscr{A}=\{Q_i\}_{i=1}^{d_n}$ is not contained in any algebraic surface in $\mathbf{P}_n^{(3)}$ (we call $g(X)=0$ an algebraic surface in $\mathbf{P}_n^{(3)}$ if $g(X)\in\mathbf{P}_n^{(3)}$ and $g(X)\not\equiv 0$).*

Definition 2. Let k be a natural number,

$$d_n(k) = \binom{n+3}{3} - \binom{n+3-k}{3} = \begin{cases} \frac{1}{6}(n+1)(n+2)(n+3), & n<k, \\ \frac{1}{6}k(3n(n-k)+12n+k^2-6k+11), & n\geqslant k \end{cases} \tag{2}$$

and $q(X)=0$ be an algebraic surface of degree k without multiple factors (or ASWMF, for short). Also suppose that $\mathscr{B}=\{Q_i\}_{i=1}^{d_n(k)}$ is a set of $d_n(k)$ distinct points on the surface $q(X)=0$. Given any set $\{f_i\}_{i=1}^{d_n(k)}$ of real numbers, we seek a polynomial $g(X)\in\mathbf{P}_n^{(3)}$ satisfying

$$g(Q_i) = f_i, \quad i=1,\ldots,d_n(k). \tag{3}$$

We call the set $\mathscr{B}=\{Q_i\}_{i=1}^{d_n(k)}$ of nodes a PPSN for polynomial interpolation of degree n along the ASWMF $q(X)=0$ of degree k and write $\mathscr{B}\in I_n^{(3)}(q)$ (where $I_n^{(3)}(q)$ denotes the set of all the PPSN for polynomial interpolation of degree n along the ASWMF $q(X)=0$), if for each given set $\{f_i\}_{i=1}^{d_n(k)}$ of real numbers there always exists a solution for the equation system (3).

Theorem B (*Liang et al.* [8]). *Let the set* $\mathscr{A}=\{Q_i\}_{i=1}^{d_n}$ *of nodes be a PPSN for* $\mathbf{P}_n^{(3)}$. *If none of these points is on an ASWMF* $q(X)=0$ *of degree* k, *then for any* $\mathscr{B}\in I_{n+k}^{(3)}(q)$, $\mathscr{B}\cup\mathscr{A}$ *must be a PPSN for* $\mathbf{P}_{n+k}^{(3)}$.

We call the constructive method given in Theorem B an *Algebraic Surface-Superposition Process* of constructing PPSN for trivariate Lagrange interpolation.

Theorem C (*Liang et al.* [8]). *Suppose an ASWMF* $q(X)=0$ *of degree* k *and a plane* $h(X)=0$ *meet at a plane algebraic curve* $C=s(q,h)$ (*where* $s(q,h)$ *is the set of common intersection-points of the algebraic surface* $q(X)=0$ *and the plane* $h(X)=0$). *If* $\mathscr{A}=\{Q_i\}_{i=1}^{d_n(k)}\in I_n^{(3)}(q)$ *and no point in* $\mathscr{A}$ *is contained in the curve* $C=S(q,h)$, *and* $\mathscr{B}\in I_{n+1}^{(3)}(C)$, *then*

$$\mathscr{A}\cup\mathscr{B}\in I_{n+1}^{(3)}(q).$$

We call the constructive method given in Theorem C a *Plane-Superposition Process* of constructing PPSN along ASWMF.

Our research is the continuation and advance of the previous work. This paper is organized as follows. In Section 2 we discuss some problems about trivariate Lagrange interpolation including Lagrange interpolation along an ASWMF and Lagrange interpolation along a space algebraic curve. In Section 3 we generalize the famous Cayley–Bacharach theorem in algebraic geometry from $\mathbf{R}^2$ to $\mathbf{R}^3$. Finally, in Section 4 we apply the generalized Cayley–Bacharach theorem in $\mathbf{R}^3$ to trivariate Lagrange interpolation and deduce several corollaries which are convenient to use.

2. Trivariate Lagrange interpolation

Firstly, we give the concepts of sufficiently intersecting of two surfaces and Lagrange interpolation along a space algebraic curve.

Definition 3. Let l and k be natural numbers. We say that an algebraic surface $p(X)=0$ of degree l and another $q(X)=0$ of degree k sufficiently intersect in a space algebraic curve $C=s(p,q)$, if there exists a plane $h(X)=0$ such that it meets the space algebraic curve $C=s(p,q)$ exactly at lk distinct points.

Definition 4. Let n, k and l be nonnegative integers, and the two algebraic surfaces $p(X)=0$ of degree l and $q(X)=0$ of degree k sufficiently intersect in a space algebraic curve $C=s(p,q)$.

$$d_n(l,k)=\binom{n+3}{3}-\binom{n-l+3}{3}-\binom{n-k+3}{3}+\binom{n-l-k+3}{3}. \tag{4}$$

Also suppose that $\mathscr{B}=\{Q_i\}_{i=1}^{d_n(l,k)}$ is a set of $d_n(l,k)$ distinct points in the curve $C=s(p,q)$. Given any set $\{f_i\}_{i=1}^{d_n(l,k)}$ of real numbers, we seek a polynomial $g(X)\in\mathbf{P}_n^{(3)}$ satisfying

$$g(Q_i)=f_i,\quad i=1,\ldots,d_n(l,k). \tag{5}$$

We call the set $\mathscr{B}=\{Q_i\}_{i=1}^{d_n(l,k)}$ of nodes a PPSN for polynomial interpolation of degree n along the curve $C=s(p,q)$ and write $\mathscr{B}\in I_n^{(3)}(C)$ (where $I_n^{(3)}(C)$ denotes the set of all the PPSN for polynomial interpolation of degree n along the curve $C=s(p,q)$), if for each given set $\{f_i\}_{i=1}^{d_n(l,k)}$ of real numbers there always exists a solution for the equation system (5).

Following Refs. [1,2], we have:

Remark 1. In Definition 2 the condition "If for each given set $\{f_i\}_{i=1}^{d_n(k)}$ of real numbers, there always exists a polynomial $g(X)\in\mathbf{P}_n^{(3)}$ satisfying $g(Q_i)=f_i$, $i=1,\ldots,d_n(k)$." can be replaced by the condition "If the relations $g(X)\in\mathbf{P}_n^{(3)}$ and $g(Q_i)=0$, $i=1,\ldots,d_n(k)$ are valid, then $g(X)\equiv 0$ along the ASWMF $q(X)=0$."

Remark 2. In Definition 4 the condition "If for each given set $\{f_i\}_{i=1}^{d_n(l,k)}$ of real numbers, there always exists a polynomial $g(X)\in\mathbf{P}_n^{(3)}$ satisfying $g(Q_i)=f_i$, $i=1,\ldots,d_n(l,k)$." can be replaced by the condition "If the relations $g(X)\in\mathbf{P}_n^{(3)}$ and $g(Q_i)=0$, $i=1,\ldots,d_n(l,k)$ are valid, then $g(X)\equiv 0$ along the curve $C=s(p,q)$."

Our main results in this paper are as follows:

Theorem 1. *Let m,n and k be natural numbers, and an ASWMF $q(X)=0$ of degree k and another $p(X)=0$ of degree m sufficiently intersect in a space algebraic curve $C=s(p,q)$. We take a PPSN $\mathscr{A}$ of degree n along the surface $q(X)=0$, i.e. $\mathscr{A}\in I_n^{(3)}(q)$ and suppose that no point in $\mathscr{A}$ is contained in $C=s(p,q)$. Moreover, we take an arbitrary PPSN $\mathscr{B}$ of degree $n+m$ along the curve $C=s(p,q)$, i.e. $\mathscr{B}\in I_{n+m}^{(3)}(C)$. Then we have*

$$\mathscr{A}\bigcup\mathscr{B}\in I_{n+m}^{(3)}(q).$$

Theorem 1 can be explained as a *Space Algebraic Curve-Superposition Process* of constructing PPSN for Lagrange interpolation along an algebraic surface, which generalizes the main result in [8].

Theorem 2. *Let m,k,l and n be natural numbers, and an algebraic surface $q(X)=0$ of degree k and another $p(X)=0$ of degree m sufficiently intersect in a space algebraic curve $C=s(p,q)$. An algebraic surface $r(X)=0$ of degree l meets the curve $C=s(p,q)$ exactly at mkl distinct points $\mathscr{A}=\{Q_i\}_{i=1}^{mkl}$. If $\mathscr{B}\in I_n^{(3)}(C)(n\geqslant m+k-3)$ and $\mathscr{B}\cap\mathscr{A}=\emptyset$, then we have*

$$\mathscr{B}\bigcup\mathscr{A}\in I_{n+l}^{(3)}(C).$$

Theorem 2 can be explained as an *Algebraic Surface-Superposition Process* of constructing PPSN for Lagrange interpolation along a space algebraic curve.

In order to prove Theorems 1 and 2, we need the following lemmas:

Lemma 1 (*Liang et al. [8]*). *Let $d_n(k)$ be defined as in* (2), *and $\mathscr{A}=\{Q_i\}_{i=1}^{d_n(k)}$ be a set of $d_n(k)$ distinct points on the ASWMF $q(X)=0$ of degree k. Then $\mathscr{A}=\{Q_i\}_{i=1}^{d_n(k)}\in I_n^{(3)}(q)$, if and only if, for any* *polynomial* $g(X) \in \mathbf{P}_n^{(3)}$ *which satisfies the zero-interpolation condition*

$$g(Q_i) = 0, \quad i = 1, \ldots, d_n(k),$$

there always exists a polynomial $r(X) \in \mathbf{P}_{n-k}^{(3)}(n \geqslant k)$ *such that*

$$g(X) = q(X)r(X).$$

(*When* $n < k$, *it means* $r(X) \equiv 0$).

Lemma 2. *Let* $d_n(l,k)$ *be defined as in* (4), *an algebraic surface* $p(X) = 0$ *of degree* l *and another* $q(X) = 0$ *of degree* k *sufficiently intersect in a space algebraic curve* $C = s(p,q)$, *and* $\mathscr{A} = \{Q_i\}_{i=1}^{d_n(l,k)}$ *be a set of* $d_n(l,k)$ *distinct points on the curve* $C = s(p,q)$. *Then* $\mathscr{A} = \{Q_i\}_{i=1}^{d_n(l,k)} \in I_n^{(3)}(C)$, *if and only if, for any polynomial* $g(X) \in \mathbf{P}_n^{(3)}$ *which satisfies the zero-interpolation condition*

$$g(Q_i) = 0, \quad i = 1, \ldots, d_n(l,k),$$

there always exist polynomials $\alpha(X) \in \mathbf{P}_{n-l}^{(3)}(n \geqslant l)$ *and* $\beta(X) \in \mathbf{P}_{n-k}^{(3)}(n \geqslant k)$ *such that*

$$g(X) = \alpha(X)p(X) + \beta(X)q(X).$$

(*Where* $n < l$ *means* $\alpha(X) \equiv 0$, *and* $n < k$ *means* $\beta(X) \equiv 0$).

Proof. *Sufficiency*: When $n < l$, by Lemma 1 the sufficiency obviously holds, so we only prove the case of $n \geqslant l$.

We choose freely a PPSN $\mathscr{B} = \{Q_i\}_{i=d_n(l,k)+1}^{d_n(l,k)+d_{n-l}(k)}$ of degree $n-l$ along the surface $q(X) = 0$, i.e. $\mathscr{B} \in I_{n-l}^{(3)}(q)$, and suppose that any point in $\mathscr{A}$ is not contained in the curve $C = s(p,q)$. Then $\mathscr{A} \cap \mathscr{B} = \emptyset$. We can prove $\mathscr{A} \cup \mathscr{B} \in I_n^{(3)}(q)$ as follows:

Firstly by Definitions 2 and 4, we know that the number of points in $\mathscr{A} \cup \mathscr{B}$ is exactly equal to the number of the points contained in a PPSN of degree n along the surface $q(X) = 0$.

Given any polynomial $g(X) \in \mathbf{P}_n^{(3)}$ satisfying the zero-interpolation condition

$$g(Q_i) = 0 \quad \forall Q_i \in \mathscr{A} \cup \mathscr{B},$$

by hypothetical condition we have

$$g(X) = \alpha(X)p(X) + \beta(X)q(X),$$

where $\alpha(X) \in \mathbf{P}_{n-l}^{(3)}$ and $\beta(X) \in \mathbf{P}_{n-k}^{(3)}$. Specially for any $Q_i \in \mathscr{B}$ we have

$$g(Q_i) = \alpha(Q_i)p(Q_i) = 0 \quad \forall Q_i \in \mathscr{B}.$$

Since $p(Q_i) \neq 0\ \forall Q_i \in \mathscr{B}$, we have $\alpha(Q_i) = 0\ \forall Q_i \in \mathscr{B}$. By Remark 1 we know $\alpha(X) \equiv 0$ along the surface $q(X) = 0$. Then $g(X) \equiv 0$ along the surface $q(X) = 0$. Using Remark 1 again we get $\mathscr{A} \cup \mathscr{B} \in I_n^{(3)}(q)$. Thus by Definition 2, for any given set $\{f_i\}_{i=1}^{d_n(k)}$ of real numbers there always exists a polynomial $g(X) \in \mathbf{P}_n^{(3)}$ satisfying

$$g(Q_i) = f_i, \quad i = 1, \ldots, d_n(l,k),$$

and

$$g(Q_i) = f_i, \quad i = d_n(l,k)+1, \ldots, d_n(k).$$

Then by Definition 4 we know $\mathscr{A} = \{Q_i\}_{i=1}^{d_n(l,k)} \in I_n^{(3)}(C)$. The sufficiency is proved.

Necessity: When $n < l$, by Definitions 4 and 2 we have $d_n(l,k) = d_n(k)$ and $\mathscr{A} \in I_n^{(3)}(q)$. Thus by Lemma 1 the necessity obviously holds, so we only prove the case of $n \geqslant l$. Firstly by the method given in Theorem C, we can constitute a PPSN $\mathscr{B} = \{Q_i\}_{i=1}^{d_{n-l}(k)}$ for polynomial interpolation of degree $n-l$ along the surface $q(X) = 0$ (where $d_{n-l}(k)$ is defined as in (2)), and suppose that any point in $\mathscr{B}$ is not contained in the curve $C = s(p,q)$. Then $\mathscr{B} \cap \mathscr{A} = \emptyset$. We can prove $\mathscr{B} \cup \mathscr{A} \in I_n^{(3)}(q)$ as follows: By Definitions 4 and 2 we know that the number of points in $\mathscr{B} \cup \mathscr{A}$ is exactly equal to the number of points contained in a PPSN of degree n along $C = s(p,q)$. Given any set $\{f_i\}_{i=1}^{d_{n-l}(k)+d_n(l,k)}$ of real numbers, since $\mathscr{A} = \{Q_i\}_{i=1}^{d_n(l,k)} \in I_n^{(3)}(C)$, by Definition 4, for the set $\{f_i\}_{i=1}^{d_n(l,k)} \subset \{f_i\}_{i=1}^{d_n(l,k)+d_{n-l}(k)}$, there always exists a polynomial $\tilde{g}(X) \in \mathbf{P}_n^{(3)}$ satisfying the following interpolation condition:

$$\tilde{g}(Q_i) = f_i, \quad i = 1, \ldots, d_n(l,k).$$

Now we construct a polynomial $\hat{g}(X)$ as follows:

$$\hat{g}(X) = \tilde{g}(X) + p(X)r(X),$$

where $r(X) \in \mathbf{P}_{n-l}^{(3)}$ and satisfies

$$\hat{g}(Q_i) = \tilde{g}(Q_i) + p(Q_i)r(Q_i) \quad \forall Q_i \in \mathscr{B}.$$

That is to say

$$r(Q_i) = (\hat{g}(Q_i) - \tilde{g}(Q_i))/p(Q_i) \quad \forall Q_i \in \mathscr{B}. \tag{6}$$

Since $\mathscr{B} \in I_{n-l}^{(3)}(q)$, $r(X) \in \mathbf{P}_{n-l}^{(3)}$, and $p(Q_i) \neq 0$ for any $Q_i \in \mathscr{B}$, then by Definition 2, there must exist a polynomial $r(X)$ satisfying the interpolation condition (6). It means that for the set $\{f_i\}_{i=d_n(l,k)+1}^{d_n(l,k)+d_{n-l}(k)} \subset \{f_i\}_{i=1}^{d_n(l,k)+d_{n-l}(k)}$ there always exists a polynomial $\hat{g}(X) \in \mathbf{P}_n^{(3)}$ such that

$$\hat{g}(Q_i) = f_i, \quad i = d_n(l,k)+1, \ldots, d_n(k).$$

Synthesizing the above results and by Definition 2 we have $\mathscr{B} \cup \mathscr{A} \in I_n^{(3)}(q)$.

Furthermore, since $\mathscr{B} \in I_{n-l}^{(3)}(q)$, then for the set $\{g(Q_i)/p(Q_i)\}_{i=1}^{d_{n-l}(k)}$, there must exist a polynomial $\alpha(X) \in \mathbf{P}_{n-l}^{(3)}$ satisfying the following interpolation condition:

$$\alpha(Q_i) = \frac{g(Q_i)}{p(Q_i)} \quad \forall Q_i \in \mathscr{B}. \tag{7}$$

Next we construct a polynomial $f(X) \in \mathbf{P}_n^{(3)}$ as follows:

$$f(X) = g(X) - \alpha(X)p(X). \tag{8}$$

It is obvious that $f(Q_i)=0$, for any $Q_i \in \mathscr{B} \cup \mathscr{A}$. Also because $\mathscr{B} \cup \mathscr{A} \in I_n^{(3)}(q)$, then by Lemma 1, there exists a polynomial $\beta(X) \in \mathbf{P}_{n-k}^{(3)}$ such that

$$f(X)=\beta(X)q(X). \tag{9}$$

Combining (8) and (9), we get

$$g(X)=\alpha(X)p(x)+\beta(X)q(X),$$

where $\alpha(X) \in \mathbf{P}_{n-l}^{(3)}$ and $\beta(X) \in \mathbf{P}_{n-k}^{(3)}$. The necessity is proved. □

Lemma 3. *Let l, k and n be natural numbers, and an algebraic surface $q(X)=0$ of degree k and another $p(X)=0$ of degree l sufficiently intersect in a space algebraic curve $C=s(p,q)$. A plane $h(X)=0$ meets the curve $C=s(p,q)$ exactly at lk distinct points $\mathscr{A}=\{Q_i\}_{i=1}^{lk}$. If $\mathscr{B} \in I_n^{(3)}(C)$ and $\mathscr{A} \cap \mathscr{B}=\emptyset$, then we have*

$$\mathscr{A} \cup \mathscr{B} \in I_{n+1}^{(3)}(C).$$

Proof. We suppose that

$$h(X)=h(x,y,z)=ax+by+c-z$$

and $Q_i=(x_i,y_i,z_i)$, $i=1,\ldots,lk$ with $\{(x_i,y_i)\}_{i=1}^{lk}$ being pairwise distinct (otherwise, we can make a coordinate rotation transform).

Dividing $h(x,y,z)=ax+by+c-z$ into $p(X)=p(x,y,z)$ and $q(X)=q(x,y,z)$, respectively, we have

$$p(x,y,z)=(ax+by+c-z)\tilde{p}(x,y,z)+r_1(x,y,ax+by+c), \tag{10}$$

$$q(x,y,z)=(ax+by+c-z)\tilde{q}(x,y,z)+r_2(x,y,ax+by+c), \tag{11}$$

where $\tilde{p}(x,y,z) \in \mathbf{P}_{l-1}^{(3)}$, $\tilde{q}(x,y,z) \in \mathbf{P}_{k-1}^{(3)}$, $r_1(x,y,ax+by+c) \in \mathbf{P}_l^{(2)}$ and $r_2(x,y,ax+by+c) \in \mathbf{P}_k^{(2)}$. By (10) and (11), for any point $Q_i \in \mathscr{A}$ we have

$$\begin{array}{l} r_1(x_i,y_i,ax_i+by_i+c)=0, \\ r_2(x_i,y_i,ax_i+by_i+c)=0, \end{array} \quad i=1,\ldots,lk.$$

So we know that the algebraic curve $r_1(x,y,ax+by+c)=0$ of degree l and $r_2(x,y,ax+by+c)=0$ of degree k intersect at lk distinct points $\{(x_i,y_i)\}_{i=1}^{lk}$.

Suppose that there exists a polynomial $g(X)=g(x,y,z) \in \mathbf{P}_{n+1}^{(3)}$ satisfying

$$g(Q_i)=0 \quad \forall Q_i \in \mathscr{A} \cup \mathscr{B}.$$

Dividing $h(X)=h(x,y,z)=ax+by+c-z$ into $g(X)=g(x,y,z)$, we get

$$g(x,y,z)=(ax+by+c-z)\tilde{g}(x,y,z)+r(x,y,ax+by+c), \tag{12}$$

where $\tilde{g}(x,y,z) \in \mathbf{P}_n^{(3)}$ and $r(x,y,ax+by+c) \in \mathbf{P}_{n+1}^{(2)}$.

By (12) for any point $Q_i \in \mathscr{A}$ we have

$$r(x_i, y_i, ax_i + by_i + c) = g(x_i, y_i, z_i) - (ax_i + by_i + c - z_i)\tilde{g}(x_i, y_i, z_i) = 0, \quad i = 1, \ldots, lk.$$

Then in [10] we get

$$r(x, y, ax + by + c) = \alpha(x, y)r_1(x, y, ax + by + c) + \beta(x, y)r_2(x, y, ax + by + c), \tag{13}$$

where $\alpha(x, y) \in \mathbf{P}^{(2)}_{n+1-l}$ and $\beta(x, y) \in \mathbf{P}^{(2)}_{n+1-k}$.

Substituting (13) into (12), we have

$$\begin{aligned} g(x, y, z) = {} & (ax + by + c - z)\tilde{g}(x, y, z) + \alpha(x, y)r_1(x, y, ax + by + c) \\ & + \beta(x, y)r_2(x, y, ax + by + c). \end{aligned} \tag{14}$$

Then substituting both (10) and (11) into (14), we have

$$\begin{aligned} g(x, y, z) &= (ax + by + c - z)\tilde{g}(x, y, z) + \alpha(x, y)(p(x, y, z) - (ax + by + c - z)\tilde{p}(x, y, z)) \\ &\quad + \beta(x, y)(q(x, y, z) - (ax + by + c - z)\tilde{q}(x, y, z)) \\ &= (ax + by + c - z)\tilde{\tilde{g}}(x, y, z) + \alpha(x, y)p(x, y, z) + \beta(x, y)q(x, y, z), \end{aligned} \tag{15}$$

where $\tilde{\tilde{g}}(x, y, z) = \tilde{g}(x, y, z) - \alpha(x, y)\tilde{p}(x, y, z) - \beta(x, y)\tilde{q}(x, y, z)$ and $\tilde{\tilde{g}}(x, y, z) \in \mathbf{P}^{(3)}_n$. For any point $Q_i \in \mathscr{B}$, by (15) we get $(ax_i + by_i + c - z_i)\tilde{\tilde{g}}(x_i, y_i, z_i) = 0$. Since $ax_i + by_i + c - z_i \neq 0$, we have $\tilde{\tilde{g}}(x_i, y_i, z_i) = 0$. Also because $\mathscr{B} \in I^{(3)}_n(C)$ and $\tilde{\tilde{g}}(x, y, z) \in \mathbf{P}^{(3)}_n$, by Lemma 2 there always exist polynomials $\tilde{\alpha}(x, y, z) \in \mathbf{P}^{(3)}_{n-l}$ and $\tilde{\beta}(x, y, z) \in \mathbf{P}^{(3)}_{n-k}$ such that

$$\tilde{\tilde{g}}(x, y, z) = \tilde{\alpha}(x, y, z)p(x, y, z) + \tilde{\beta}(x, y, z)q(x, y, z). \tag{16}$$

Substituting (16) into (15), we have

$$g(x, y, z) = \alpha(x, y, z)p(x, y, z) + \beta(x, y, z)q(x, y, z),$$

where

$$\alpha(x, y, z) = (ax + by + c - z)\tilde{\alpha}(x, y, z) + \alpha(x, y),$$
$$\beta(x, y, z) = (ax + by + c - z)\tilde{\beta}(x, y, z) + \beta(x, y)$$

and $\alpha(x, y, z) \in \mathbf{P}^{(3)}_{n+1-l}$ and $\beta(x, y, z) \in \mathbf{P}^{(3)}_{n+1-k}$. By Lemma 2 we can get $\mathscr{A} \cup \mathscr{B} \in I^{(3)}_{n+1}(C)$. □

Lemma 4 (*Mysovskikh [9]*). *Let m, n and k be natural numbers, and r be an integer. If the three surfaces $p(X) = 0$ of degree m, $q(X) = 0$ of degree n and $r(X) = 0$ of degree k meet exactly at mnk distinct points, and an algebraic surface $f(X) = 0$ of degree r passes through these mnk distinct points, then there must exist polynomials $\alpha(X) \in \mathbf{P}^{(3)}_{r-m}$, $\beta(X) \in \mathbf{P}^{(3)}_{r-n}$ and $\gamma(X) \in \mathbf{P}^{(3)}_{r-k}$ such that*

$$f(X) = \alpha(X)p(X) + \beta(X)q(X) + \gamma(X)r(X).$$

Now, we begin to prove Theorems 1 and 2.

Proof of Theorem 1. By Definitions 2 and 4, we know that the number of points in $\mathscr{A} \cup \mathscr{B}$ is exactly equal to the number of points contained in a PPSN of degree $n + m$ along the surface $q(X) = 0$ of degree k.

Suppose $g(X) \in \mathbf{P}^{(3)}_{n+m}$ and $g(Q_i)=0$ for any $Q_i \in \mathscr{A} \cup \mathscr{B}$. From Lemma 1, we only need to prove that there exists a polynomial $r(X) \in \mathbf{P}^{(3)}_{m+n-k}$ such that

$$g(X) = q(X)r(X).$$

Since $g(X) \in \mathbf{P}^{(3)}_{n+m}$, $g(Q_i)=0$ for any $Q_i \in \mathscr{B}$, and $\mathscr{B} \in I^{(3)}_{n+m}(C)$, then by Lemma 2 there exist polynomials $\alpha(X) \in \mathbf{P}^{(3)}_{n}$ and $\beta(X) \in \mathbf{P}^{(3)}_{n+m-k}$ such that

$$g(X) = \alpha(X)p(X) + \beta(X)q(X). \tag{17}$$

Also since $g(Q_i)=0$ for any $Q_i \in \mathscr{A}$, from (17), we get $\alpha(Q_i)p(Q_i)=0$. But $p(Q_i) \neq 0$, so $\alpha(Q_i)=0$. Because $\mathscr{A} \in I^{(3)}_{n}(q)$ and $\alpha(X) \in \mathbf{P}^{(3)}_{n}$, then from Lemma 1 there exists a polynomial $\tilde{r}(X) \in \mathbf{P}^{(3)}_{n-k}$ such that

$$\alpha(X) = q(X)\tilde{r}(X). \tag{18}$$

Substituting (18) into (17), we get

$$g(X) = q(X)r(X),$$

where $r(X) = p(X)\tilde{r}(X) + \beta(X)$, and $r(X) \in \mathbf{P}^{(3)}_{m+n-k}$. $\square$

Proof of Theorem 2. The number of points in $\mathscr{B} \cup \mathscr{A}$ is

$$\tfrac{1}{2}mk(2n+4-m-k) + mkl = \tfrac{1}{2}mk(2(n+l)+4-m-k),$$

which is exactly equal to the number of points contained in a PPSN of degree $n+l$ along the curve $C = s(p,q)$.

Suppose $g(X) \in \mathbf{P}^{(3)}_{n+l}$ satisfies $g(Q_i)=0$ for any $Q_i \in \mathscr{A} \cup \mathscr{B}$. From Lemma 2, we only need to prove that there exist polynomials $\alpha(X) \in \mathbf{P}^{(3)}_{n+l-m}$ and $\beta(X) \in \mathbf{P}^{(3)}_{n+l-k}$ such that

$$g(X) = \alpha(X)p(X) + \beta(X)q(X).$$

Since $g(Q_i)=0$ for any $Q_i \in \mathscr{A}$, then by Lemma 4, there exist polynomials $\tilde{\alpha}(X) \in \mathbf{P}^{(3)}_{n+l-m}$, $\tilde{\beta}(X) \in \mathbf{P}^{(3)}_{n+l-k}$ and $\tilde{\gamma}(X) \in \mathbf{P}^{(3)}_{n}$ such that

$$g(X) = \tilde{\alpha}(X)p(X) + \tilde{\beta}(X)q(X) + \tilde{\gamma}(X)r(X). \tag{19}$$

Also since $g(Q_i)=0$ for any $Q_i \in \mathscr{B}$, from (19) we get $\tilde{\gamma}(Q_i)r(Q_i)=0\ \forall Q_i \in \mathscr{B}$. But $r(Q_i) \neq 0$, so $\tilde{\gamma}(Q_i)=0$. Because $\tilde{\gamma}(X) \in \mathbf{P}^{(3)}_{n}$ and $\mathscr{B} \in I^{(3)}_{n}(C)$, then from Lemma 2 there exist polynomials $\hat{\alpha}(X) \in \mathbf{P}^{(3)}_{n-m}$ and $\hat{\beta}(X) \in \mathbf{P}^{(3)}_{n-k}$ such that

$$\tilde{\gamma}(X) = \hat{\alpha}(X)p(X) + \hat{\beta}(X)q(X). \tag{20}$$

Substituting (20) into (19), we get

$$g(X) = \alpha(X)p(X) + \beta(X)q(X),$$

where $\alpha(X) = \tilde{\alpha}(X) + \hat{\alpha}(X)r(X)$, $\alpha \in \mathbf{P}^{(3)}_{n+l-m}$, $\beta(X) = \tilde{\beta}(X) + \hat{\beta}(X)r(X)$, $\beta(X) \in \mathbf{P}^{(3)}_{n+l-k}$. $\square$

Remark 3. Lemma 3 is a special case of Theorem 2 when $l=1$.

3. The generalization of Cayley–Bacharach theorem from $\mathbf{R}^2$ to $\mathbf{R}^3$

At first we introduce the famous Cayley–Bacharach theorem in $\mathbf{R}^2$.

Theorem E (*Semple and Roth [10]*). *Let m, n and r be natural numbers, and $3 \leqslant r \leqslant \min\{m, n\} + 2$. Suppose that the two algebraic curves $p(x, y) = 0$ of degree m and $q(x, y) = 0$ of degree n meet exactly at mn distinct points. If $f(x, y) \in \mathbf{P}^{(2)}_{m+n-r}$ (where $\mathbf{P}^{(2)}_n$ denotes the space of all real bivariate polynomials of total degree $\leqslant n$) and the algebraic curve $f(x, y) = 0$ passes through $mn - \frac{1}{2}(r-1)(r-2)$ points of those mn points, then it must pass through the $\frac{1}{2}(r-1)(r-2)$ remainder points, unless these $\frac{1}{2}(r-1)(r-2)$ remainder points lie on one curve of degree $r - 3$.*

In order to obtain another very important result about Lagrange interpolation along a space algebraic curve in this paper, we need expand the above theorem to the case in $\mathbf{R}^3$.

Theorem 3. *Let m, n, k and r be natural numbers, and $4 \leqslant r \leqslant \min\{m, n, k\} + 3$. Suppose that the three algebraic surfaces $p(X) = 0$ of degree m, $q(X) = 0$ of degree n and $r(X) = 0$ of degree k meet exactly at mnk distinct points. If there exists a polynomial $f(X) \in \mathbf{P}^{(3)}_{m+n+k-r}$ such that the surface $f(X) = 0$ passes through $mnk - \frac{1}{6}(r-1)(r-2)(r-3)$ points of those mnk points, then it must pass through the $\frac{1}{6}(r-1)(r-2)(r-3)$ remainder points, unless these $\frac{1}{6}(r-1)(r-2)(r-3)$ remainder points lie on one algebraic surface of degree $r - 4$.*

Proof. We prove the theorem in two cases: (i) $r = 4$; (ii) $r > 4$.

(i) For $r = 4$, we can get the conclusion of Theorem 3 by the Corollary G in [11].
(ii) For $r > 4$, let $d = \frac{1}{6}(r-1)(r-2)(r-3)$. $\hat{\mathscr{A}} = \{Q_i\}_{i=1}^d$ denotes the set of the remainder points. Since $\hat{\mathscr{A}} = \{Q_i\}_{i=1}^d$ does not lie on any algebraic surface of degree $r - 4$, then by Theorem A $\hat{\mathscr{A}} = \{Q_i\}_{i=1}^d$ must constitute a PPSN for $\mathbf{P}^{(3)}_{r-4}$.

Let $\{l_i(X) \in \mathbf{P}^{(3)}_{r-4}\}_{i=1}^d$ be the corresponding basic polynomials of Lagrange interpolation, then $l_i(X)f(X) \in \mathbf{P}^{(3)}_{m+n+k-4}$ and the surface $l_i(X)f(X) = 0$ passes through all points in $\mathscr{A}$ but Q_i. By the proof of the case (i), we know that the surface $l_i(X)f(X) = 0$ must pass through all mnk points of intersection. It follows from Lemma 4 that there exist polynomials $\alpha(X) \in \mathbf{P}^{(3)}_{n+k-4}$, $\beta(X) \in \mathbf{P}^{(3)}_{m+k-4}$ and $\gamma(X) \in \mathbf{P}^{(3)}_{m+n-4}$ such that

$$l_i(X)f(X) = \alpha(X)p(X) + \beta(X)q(X) + \gamma(X)r(X).$$

Hence for any $Q_i \in \mathscr{A}$, we have $l_i(Q_i)f(Q_i) = 0$. But $l_i(Q_i) = 1 \neq 0$, so $f(Q_i) = 0$. Taking $i = 1, \ldots, mnk$, we get the conclusion of the theorem. $\square$

4. The application of the generalized Cayley–Bacharach theorem to Lagrange interpolation along a space algebraic curve.

Using Theorem 3 we can get the following important result:

Theorem 4. *Let $\{0\} = \mathbf{P}^{(3)}_{-1} = \mathbf{P}^{(3)}_{-2} = \cdots$, denote the space of zero polynomials, and under these circumstances we regard their corresponding PPSN as the empty set. Suppose m, n and k are natural numbers, $m \leqslant n \leqslant k$, and σ is an integer number satisfying $\sigma \geqslant 1 - m$. Also suppose that the three algebraic surfaces $p(X) = 0$ of degree m, $q(X) = 0$ of degree n and $r(X) = 0$ of degree k meet exactly at mnk distinct points $\mathscr{A} = \{Q_i\}_{i=1}^{mnk}$, and $\mathscr{B} \subseteq \mathscr{A}$ is a PPSN for $\mathbf{P}^{(3)}_{-\sigma}$. Then we have*

(1) *If $\mathscr{A}_1$ is a PPSN for polynomial interpolation of degree $m + n + \sigma - 4$ along the space algebraic curve $C_1 = s(p, q)$, and $\mathscr{A}_1 \cap \mathscr{A} = \emptyset$, then $\mathscr{A}_1 \cup (\mathscr{A} \backslash \mathscr{B})$ must be a PPSN for polynomial interpolation of degree $m + n + k + \sigma - 4$ along the space algebraic curve $C_1 = s(p, q)$;*
(2) *If $\mathscr{A}_2$ is a PPSN for polynomial interpolation of degree $m + k + \sigma - 4$ along the space algebraic curve $C_2 = s(p, r)$, and $\mathscr{A}_2 \cap \mathscr{A} = \emptyset$, then $\mathscr{A}_2 \cup (\mathscr{A} \backslash \mathscr{B})$ must be a PPSN for polynomial interpolation of degree $m + n + k + \sigma - 4$ along the space algebraic curve $C_2 = s(p, r)$;*
(3) *If $\mathscr{A}_3$ is a PPSN for polynomial interpolation of degree $n + k + \sigma - 4$ along the space algebraic curve $C_3 = s(q, r)$, and $\mathscr{A}_3 \cap \mathscr{A} = \emptyset$, then $\mathscr{A}_3 \cup (\mathscr{A} \backslash \mathscr{B})$ must be a PPSN for polynomial interpolation of degree $m + n + k + \sigma - 4$ along the space algebraic curve $C_3 = s(q, r)$.*

Remark 4. In conclusion (1) of Theorem 4, when $m + n + \sigma - 4 < 0$ we regard the empty set as the PPSN of degree $m + n + \sigma - 4$ along the space algebraic curve C_1. We have the similar results to (2) and (3).

Proof of Theorem 4. Since the proofs of those conclusions are similar, we only prove conclusion (1) here. We consider three cases: $\sigma \geqslant 4$, $\sigma < 0$ and $\sigma = 0, 1, 2, 3$.

(1) For $\sigma \geqslant 4$, in this case we have $\mathscr{B} = \phi$. By Definition 4, we know that the number of points in $\mathscr{A}_1 \cup (\mathscr{A} \backslash \mathscr{B}) = \mathscr{A}_1 \cup \mathscr{A}$ is

$$\tfrac{1}{2}mn(m + n + 2\sigma - 4) + mnk = \tfrac{1}{2}mn(m + n + 2\sigma + 2k - 4).$$

The number of points contained in a PPSN for polynomial interpolation of degree $m + n + k + \sigma - 4$ along the curve $C_1 = s(p, q)$ is

$$\tfrac{1}{2}mn(m + n + 2\sigma + 2k - 4).$$

It is obvious that the number of points in $\mathscr{A}_1 \cup \mathscr{A}$ is exactly equal to the number of points contained in a PPSN of degree $m + n + k + \sigma - 4$ along the curve $C_1 = s(p, q)$.

Let $g(X) \in \mathbf{P}^{(3)}_{m+n+k+\sigma-4}$ satisfy

$$g(Q_i) = 0 \quad \forall Q_i \in \mathscr{A}_1 \bigcup \mathscr{A}.$$

Since $g(Q_i) = 0$ for every $Q_i \in \mathscr{A}$, from Lemma 4 there exist polynomials $\alpha(X) \in \mathbf{P}^{(3)}_{n+k+\sigma-4}$, $\beta(X) \in \mathbf{P}^{(3)}_{m+k+\sigma-4}$ and $\gamma(X) \in \mathbf{P}^{(3)}_{m+n+\sigma-4}$ such that

$$g(X) = \alpha(X)p(X) + \beta(X)q(X) + \gamma(X)r(X). \tag{21}$$

Since $g(Q_i) = 0$ for any $Q_i \in \mathscr{A}_1$, by (21) we have $\gamma(Q_i)r(Q_i) = 0$. But $r(Q_i) \neq 0$ for any $Q_i \in \mathscr{A}$, so we have $\gamma(Q_i) = 0$. Also because $\gamma(X) \in \mathbf{P}^{(3)}_{m+n+\sigma-4}$ and $\mathscr{A}_1 \in I^{(3)}_{m+n+\sigma-4}(C_1)$, from Lemma 2 we know there exist polynomials $\tilde{\alpha}(X) \in \mathbf{P}^{(3)}_{n+\sigma-4}$ and $\tilde{\beta}(X) \in \mathbf{P}^{(3)}_{m+\sigma-4}$ such that

$$\gamma(X) = \tilde{\alpha}(X)p(X) + \tilde{\beta}(X)q(X). \tag{22}$$

Substituting (22) to (21), we have

$$g(X) = \hat{\alpha}(X)p(X) + \hat{\beta}(X)q(X),$$

where $\hat{\alpha}(X) = \alpha(X) + \tilde{\alpha}(X)r(X)$, $\hat{\alpha}(X) \in \mathbf{P}^{(3)}_{n+k+\sigma-4}$ and $\hat{\beta}(X) = \beta(X) + \tilde{\beta}(X)r(X)$, $\hat{\beta}(X) \in \mathbf{p}^{(3)}_{m+k+\sigma-4}$. Then it follows from Lemma 2 that

$$\mathscr{A}_1 \cup \mathscr{A} = \mathscr{A}_1 \cup (\mathscr{A}\backslash\mathscr{B}) \in I^{(3)}_{m+n+k+\sigma-4}(C_1).$$

(2) For $\sigma < 0$, according to Definitions 1 and 4 and by a simple calculation, we know that the number of points in $\mathscr{A}_1 \cup (\mathscr{A}\backslash\mathscr{B})$ is exactly equal to the number of points contained in a PPSN of degree $m+n+k+\sigma-4$ along the curve $C_1 = s(p, q)$.

Suppose $g(X) \in \mathbf{P}^{(3)}_{m+n+k+\sigma-4}$ satisfies $g(Q_i) = 0$ for every $Q_i \in \mathscr{A}_1 \cup (\mathscr{A}\backslash\mathscr{B})$. Since $\mathscr{B}$ is a PPSN for $\mathbf{P}^{(3)}_{-\sigma}$, from Theorem A $\mathscr{B}$ is not contained in any algebraic surface of degree $-\sigma$. Also since $g(Q_i) = 0$ for any $Q_i \in \mathscr{A}\backslash\mathscr{B}$, from Theorem 3 we have $g(Q_i) = 0$ for any $Q_i \in \mathscr{A}$. By Lemma 4 there exist polynomials $\alpha(X) \in \mathbf{P}^{(3)}_{n+k+\sigma-4}$, $\beta(X) \in \mathbf{P}^{(3)}_{m+k+\sigma-4}$ and $\gamma(X) \in \mathbf{P}^{(3)}_{m+n+\sigma-4}$ such that

$$g(X) = \alpha(X)p(X) + \beta(X)q(X) + \gamma(X)r(X). \tag{23}$$

Since $g(Q_i) = 0$ for any $Q_i \in \mathscr{A}_1$, from (23) we have $\gamma(Q_i)r(Q_i) = 0$. But $r(Q_i) \neq 0$ for any $Q_i \in \mathscr{A}_1$, so we have $\gamma(Q_i) = 0$. Also because $\mathscr{A}_1 \in I^{(3)}_{m+n+\sigma-4}(C_1)$ and $\gamma(X) \in \mathbf{P}^{(3)}_{m+n+\sigma-4}$, therefore from Remark 2 we have $\gamma(X) \equiv 0$ along the curve $C_1 = s(p, q)$. From (23) we have $g(X) \equiv 0$ along the curve $C_1 = s(p, q)$. Then it follows from Remark 2 that $\mathscr{A}_1 \cup (\mathscr{A}\backslash\mathscr{B}) \in I^{(3)}_{m+n+k+\sigma-4}(C_1)$.

For $\sigma = 0, 1, 2, 3$, we can complete the proof just like that in case (1) and (2), so we omit it here. □

For Theorem 4, we deduce the following three corollaries which are convenient to use.

Let $\sigma \leqslant 0$ and $r = 4 - \sigma$ in Theorem 4, then we have:

Corollary 1. *Suppose that m, n and k are natural numbers, $m \leqslant n \leqslant k$, r is a nonnegative integer, and $4 \leqslant r \leqslant m+3$. If the three algebraic surfaces $p(X) = 0$ of degree m, $q(X) = 0$ of degree n and $r(X) = 0$ of degree k meet exactly at mnk distinct points $\mathscr{A} = \{Q_i\}_{i=1}^{mnk}$, and $\mathscr{B} \subset \mathscr{A}$ is a PPSN for $\mathbf{P}^{(3)}_{r-4}$, then we have*

(1) *If $\mathscr{D}_1$ is a PPSN for polynomial interpolation of degree $m+n-r$ along the space algebraic curve $C_1 = s(p, q)$, and $\mathscr{D}_1 \cap \mathscr{A} = \emptyset$, then $\mathscr{D}_1 \cup (\mathscr{A}\backslash\mathscr{B})$ must constitute a PPSN for polynomial interpolation of degree $m+n+k-r$ along the curve $C_1 = s(p, q)$;*
(2) *If $\mathscr{D}_2$ is a PPSN for polynomial interpolation of degree $n+k-r$ along the space algebraic curve $C_2 = s(q, r)$, and $\mathscr{D}_2 \cap \mathscr{A} = \emptyset$, then $\mathscr{D}_2 \cup (\mathscr{A}\backslash\mathscr{B})$ must constitute a PPSN for polynomial interpolation of degree $m+n+k-r$ along the curve $C_2 = s(q, r)$;*
(3) *If $\mathscr{D}_3$ is a PPSN for polynomial interpolation of degree $m+k-r$ along the space algebraic curve $C_3 = s(p, r)$, and $\mathscr{D}_3 \cap \mathscr{A} = \emptyset$, then $\mathscr{D}_3 \cup (\mathscr{A}\backslash\mathscr{B})$ must constitute a PPSN for polynomial interpolation of degree $m+n+k-r$ along the curve $C_3 = s(p, r)$.*

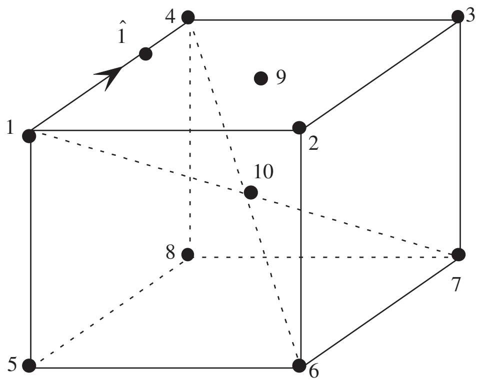

Fig. 1.

Let $\sigma \geqslant 1$ and $r = \sigma - 1$ in Theorem 4, then we have:

Corollary 2. *Suppose that m, n and k are natural numbers, and r is a nonnegative integer. If the three algebraic surfaces $p(X) = 0$ of degree m, $q(X) = 0$ of degree n and $r(X) = 0$ of degree k meet exactly at mnk distinct points $\mathscr{A} = \{Q_i\}_{i=1}^{mnk}$, then we have*

(1) *If $\mathscr{E}_1$ is a PPSN for polynomial interpolation of degree $m+n+r-3$ along the space algebraic curve $C_1 = s(p, q)$, and $\mathscr{E}_1 \cap \mathscr{A} = \emptyset$, then $\mathscr{E}_1 \cup \mathscr{A}$ must constitute a PPSN for polynomial interpolation of degree $m+n+k+r-3$ along the curve $C_1 = s(p, q)$;*
(2) *If $\mathscr{E}_2$ is a PPSN for polynomial interpolation of degree $n+k+r-3$ along the space algebraic curve $C_2 = s(q, r)$, and $\mathscr{E}_2 \cap \mathscr{A} = \emptyset$, then $\mathscr{E}_2 \cup \mathscr{A}$ must constitute a PPSN for polynomial interpolation of degree $m+n+k+r-3$ along the curve $C_2 = s(q, r)$;*
(3) *If $\mathscr{E}_3$ is a PPSN for polynomial interpolation of degree $m+k+r-3$ along the space algebraic curve $C_3 = s(p, r)$, and $\mathscr{E}_3 \cap \mathscr{A} = \emptyset$, then $\mathscr{E}_3 \cup \mathscr{A}$ must constitute a PPSN for polynomial interpolation of degree $m+n+k+r-3$ along the curve $C_3 = s(p, r)$.*

Furthermore let $\sigma \geqslant 1$ and $r = m + n + \sigma - 4$ (or $r = m + k + \sigma - 4$, or $r = n + k + \sigma - 4$) in Theorem 4, then we have:

Corollary 3. *Suppose that m, n and k are natural numbers, and r is a nonnegative integer. If the three algebraic surfaces $p(X) = 0$ of degree m, $q(X) = 0$ of degree n and another $r(X) = 0$ of degree k meet exactly at mnk distinct points $\mathscr{A} = \{Q_i\}_{i=1}^{mnk}$, then we have*

(1) *If $\mathscr{F}_1 \in I_r(C_1 = s(p, q))$ $(r \geqslant m + n - 3)$ and $\mathscr{F}_1 \cap \mathscr{A} = \emptyset$, then $\mathscr{F}_1 \cup \mathscr{A} \in I_{r+k}(C_1 = s(p, q))$;*
(2) *If $\mathscr{F}_2 \in I_r(C_2 = s(q, r))$ $(r \geqslant n + k - 3)$ and $\mathscr{F}_2 \cap \mathscr{A} = \emptyset$, then $\mathscr{F}_2 \cup \mathscr{A} \in I_{r+m}(C_2 = s(q, r))$;*
(3) *If $\mathscr{F}_3 \in I_r(C_3 = s(p, r))$ $(r \geqslant m + k - 3)$ and $\mathscr{F}_3 \cap \mathscr{A} = \emptyset$, then $\mathscr{F}_3 \cup \mathscr{A} \in I_{r+n}(C_3 = s(p, r))$.*

Now, in order to explain our constructing theorems, we give a example (see Fig. 1): Suppose that three algebraic surfaces $p(X) = h_{1234}(X)h_{5678}(X) = 0$, $q(X) = h_{1458}(X)h_{2367}(X) = 0$ and $r(X) = h_{1256}(X)h_{3478}(X) = 0$ (where $h_{abcd}(X) = 0$ denotes the plane equation passing points a, b, c and d) of the same degree 2 meet exactly at 8 distinct points $\mathscr{A} = \{1, 2, ..., 8\}$, then these points do not constitute a PPSN for any subspace of $\mathbf{P}_2^{(3)}$. But if we move any point of these along the space algebraic curve $C_1 = s(p, q) = l_{14}(X)l_{23}(X)l_{58}(X)l_{67}(X) = 0$ (where $l_{ab}(X) = 0$ denotes the straight line equation passing points a and b) to a new position (e.g. move 1 to $\hat{1}$), then by Theorem 4 these current eight points $\{\hat{1}, 2, \ldots, 8\}$ must constitute a PPSN for polynomial interpolation of degree 2 along the curve $C_1 = s(p, q)$. If we choose any point (e.g. point 9) on the surface $p(X) = 0$ without points $\{\hat{1}\} \cup \mathscr{A}$, then by Theorem 1 the points $\{\hat{1}, 2, \ldots, 8, 9\}$ must constitute a PPSN of degree 2 along the surface $p(X) = 0$. Furthermore, if we choose any point (e.g. point 10) in $\mathbf{R}^3$ without the surface $p(X) = 0$, then by Theorem B the points $\{\hat{1}, 2, \ldots, 10\}$ must constitute a PPSN for $\mathbf{P}_2^{(3)}$.

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